

EE 216 - Experiment 11
The Laplace Transform and Control System Characteristics
Objectives:

To illustrate computer usage in determining inverse Laplace transforms. Also to determine useful signal and system characteristics by using Laplace transform theory and computer calculations.

Theory:

The Laplace transform is a convenient tool in the analysis of linear time-invariant systems. In particular, we can transform the system set of integro-differential equations or the system representation (component or block diagram). This permits us to find algebraic transformed equations that represent the system. We use these algebraic equations to solve for the transform of the output signal $Y(s)$ after we have computed the transform of the input signal $X(s)$ and supplied the initial conditions. The algebraic solution is simpler than a differential equation solution. All that remains to find the output signal at this point is to compute the inverse transform of $Y(s)$. This can easily be done for rational transforms by using partial fraction expansion, though the computations are tedious.

While there is symbolic mathematics software (e.g., MATLAB Symbolic Math Toolbox) that we can use to compute Laplace transforms of signals, it is not too effective except for transforms that are easily found in a table. Thus we will not use MATLAB to compute Laplace transforms here.

The computer is very helpful in eliminating the tedious computations required to compute partial fraction expansion (PFE) coefficients for rational transforms. These coefficients and the corresponding transform pole locations can be used to generate equations easily for the signal that corresponds to a Laplace transform. The MATLAB function that generates the PFE coefficients is **[r,p,k]=residue(n, d)**. The column arrays **r** and **p** are the PFE coefficients and the corresponding pole locations, respectively. The row array **k** contains the coefficients of the direct terms present when the transform is not proper; that is, the terms obtained by long division when the transform numerator degree is greater than or equal to the transform denominator degree. As an example, consider the transform.

$$X(s) = \frac{n_8 s^8 + n_7 s^7 + \dots + n_1 s + n_o}{d_6 s^6 + d_5 s^5 + \dots + d_1 s + d_o} \quad (11.1)$$

Let us assume that this transform becomes

$$X(s) = \frac{n_8 s^8 + n_7 s^7 + \dots + n_1 s + n_o}{s(s+a)(s+b)^2(s+c+jd)(s+c-jd)} \quad (11.2)$$

when the denominator is factored. To compute the PFE coefficients of $X(s)$, the inputs to **residue** must be the row arrays $\mathbf{n} = [n_8, n_7, \dots, n_1, n_0]$ and $\mathbf{d} = [d_6, d_5, \dots, d_1, d_0]$. The output of **residue** is then:

$\mathbf{r} =$	$\mathbf{p} =$	$\mathbf{k} =$
R_1	0	$k_0 \quad k_1$
R_2	$-a$	
R_3	$-b$	
R_4	$-b$	
$R_5 + I_5i$	$-c - di$	
$R_5 - I_5i$	$-c + di$	

corresponding to the PFE

$$X(s) = k_1 s + k_0 + \frac{R_1}{s} + \frac{R_2}{s+a} + \frac{R_3}{s+b} + \frac{R_4}{(s+b)^2} + \frac{R_5 + jI_5}{s+c+jd} + \frac{R_5 - jI_5}{s+c-jd} \quad (11.3)$$

From the tables, the signal corresponding to the PFE is

$$x(t) = k_1 \delta'(t) + k_0 \delta(t) + R_1 u(t) + R_2 e^{-at} u(t) + (R_3 + R_4 t) e^{-bt} u(t) + 2e^{-ct} [R_5 \cos(dt) + I_5 \sin(dt)] u(t) \quad (11.4)$$

Thus, we can easily write the mathematical expression for the signal and plot it once we have used **residue** to find the PFE parameters. In this experiment we restrict ourselves to proper rational functions. In this way we avoid having to approximate impulses and derivatives of impulses for plotting purposes.

We see that pole locations for a signal give us important information about the signal time waveform characteristics. The zero locations also affect the signal term multipliers (the PFE coefficients). The pole and zero locations are the roots of the numerator and denominator polynomials respectively. Therefore, we can use $\mathbf{z} = \mathbf{roots}(\mathbf{n})$ and $\mathbf{p} = \mathbf{roots}(\mathbf{d})$ to find them. Since the pole and zero locations give us useful information about a signal, we often generate a pole-zero plot to show these locations.

MATLAB has a built-in function to plot poles and zeros in the z-plane called **zplane(z,p)**. This function can also plot poles and zeros from transfer function coefficients. See the help file for details.

If a system is linear, time-invariant, and has zero initial conditions, then the output $y(t)$ is

$$y(t) = x(t) * h(t) \quad (11.5)$$

where $x(t)$ is the input and $h(t)$ is the impulse response. Using the convolution theorem, the Laplace transform of the output is

$$Y(s) = X(s)H(s) \quad (11.6)$$

We call the Laplace transform of the impulse response, $H(s)$, the system transfer function. We can find it directly from the set of integro-differential equations that characterize the system or directly from the system representation (component or block diagram). The transfer function is extremely important to us since it completely characterizes the system except for initial conditions.

We can use the function **roots** on the transfer function denominator polynomial to determine system pole locations. From these pole locations we can determine the system stability characteristics.

For a stable system, we can also find the frequency response directly from the transfer function by replacing s by $j\omega$. Therefore, the amplitude response is a slice through the transfer function amplitude surface along the imaginary axis. We demonstrate this in Part 7 in this experiment. Also, the important system parameters of break frequencies and damping ratios can be found directly from the transfer function, since its coefficients are the same as those of the frequency response. This also means that we can plot the straight-line approximation to a system's Bode amplitude and phase responses directly from the transfer function. This straight-line approximation can be used to illustrate basic amplitude and phase response characteristics and help us choose any control system compensation that may be necessary.

Preliminary:

1. Find the transfer function for the feedback control system characterized by the block diagram representation shown in Figure 11.1. You will use the transfer function in Part 6 of the experiment procedure.
2. Find the inverse Laplace transform of the equation in Part 4. HINT: use your textbook for help.

Laboratory Procedure:

1. Find and plot the mathematical expressions for the signals corresponding to the Laplace transforms

- a.
$$W(s) = \frac{3.5s^2 + 2.2s + 7.5}{5s^3 + 40.55s^2 + 82.15s + 33.75}$$

$$b. \quad Y(s) = \frac{4s^2 + 10s + 12}{s^4 + 5s^3 + 16.25s^2 + 40s + 27.75}$$

2. Use the function **zplane** to create the pole-zero plot for the Laplace transforms of Part 1.
3. Consider the simple Laplace transform.

$$X(s) = \frac{s^3 + 9s}{s^3 + 5s^2 + 11s + 15}$$

- a. Use the function **zplane** to create the pole zero plot for $X(s)$.
- b. The parameter s equals the complex variable $\sigma + j\omega$ and its values can be plotted on the s -plane which has axes of σ and ω . Thus, $|X(s)|$ is a function of the two variables σ and ω and can be plotted as a surface above the s -plane. You can use the following script to plot $|X(s)|$ over the region $-10 < \sigma < 0$, $-5 < \omega < 5$.

```
[a,w]=meshgrid(-10:0.1:0,-5:0.1:5); s=a+j*w;
X=(s.^3+9s)./(s.^3+5*s.^2+11*s+15);
surface(w,a,abs(X)); axis([-5 5 -10 0 0 6]);
view(-24,40);
set(gca,'xdir','reverse','ydir','reverse');
xlabel('w'); ylabel('a'); zlabel('|H(s)|');
```

We have plotted only for negative values of s so the zero shows more clearly.

Surface plots like the one plotted help us visualize the interaction of the poles and zeros on the transform values. A surface plot of the angle can also be plotted. It is usually not easy to interpret. Try the script on some other transforms. It is not always possible to produce the plot due to numerical computation problems.

- 4.. Find and plot the mathematical expression for the signal corresponding to the Laplace transform.

$$X(s) = \frac{(0.5s^3 + 0.375s^2 + 0.0625s) - 1.5s^2e^{-s} - (s^2 + 0.75s + 0.125)e^{-2.5s}}{s^4 + 0.75s^3 + 0.125s^2}$$

5. Plot the pole-zero plot and the signal that corresponds to the transform $X(s) = \frac{1}{s+c}$ for $c = 0.03, 0.06,$ and 0.12 . What is the effect of the pole location on the signal characteristics?
6. The transformed block diagram representation for a submarine depth control system is shown in Figure 11.1. In this block diagram, $C(s)$ is the transform of the commanded depth and $D(s)$ is the transform of the depth.

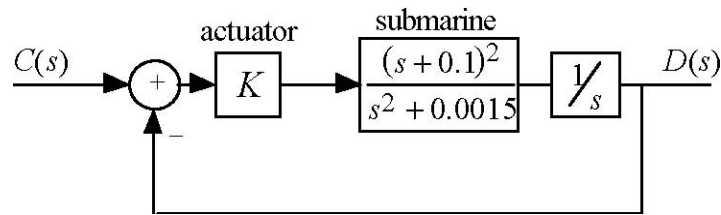


Figure 11.1 Submarine Depth Control System

You found the transfer function for this system in Part 1 of the Preliminary.

- a. A quick 50 ft dive command issued at $t = 0$ can be modeled as $c(t) = -50u(t)$. Find the resulting submarine depth as a function of time for $K = 0.4$, $K = 0.08$, and $K = 0.04$. Use a time scale interval of 0 to 30 s for $K = 0.4$ and 0 to 200 s for $K = 0.08$ and 0.04 to illustrate the significant characteristics. Describe the characteristics in terms of submarine behavior.
- b. A gentler dive command is shown in Figure 11.2

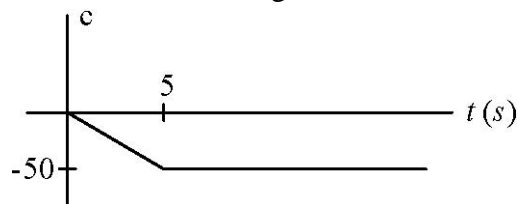


Figure 11.2 Gentler Dive Command

Repeat Part a for this dive command. Compare with the results of part a.

7. In this part of the experiment, we investigate the frequency response of fifth-order Butterworth and Chebyshev approximations to the ideal LPF. The filters are normalized to have a maximum gain of 1 and a cutoff frequency of 1 rad/s. The Butterworth filter transfer function is

$$H_b(s) = \frac{1}{s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1} \quad (11.7)$$

And the Chebyshev filter transfer function is

$$H_c(s) = \frac{0.0626}{s^5 + 0.5745s^4 + 1.4150s^3 + 0.5489s^2 + 0.4080s + 0.0626} \quad (11.8)$$

- a. Plot the pole-zero plots for each filter. Also, plot a circle of radius 1 and the ellipse specified by the equation

$$\left| \frac{\sigma}{0.177} \right|^2 + \left| \frac{\omega}{1.016} \right|^2 = 1 \quad (11.9)$$

on each plot. What do you notice about the pole locations?

- b. Plot the amplitude and phase responses for both filters for $-2 \leq \omega \leq 2$ rad/s using linear scales. Plot both amplitude responses on one set of axes and both phase responses on one set of axes.
- c. Since the frequency response is $H(s)$ with $s = j\omega$, then the amplitude response is the slice through $|H(s)|$ along the imaginary axis. To see this, use the surface plotting script from Part 3b to plot $|H(s)|$ for both filters. The near edge of the plot is along the imaginary axis. Therefore, the near edge of the plot is the slice through $|H(s)|$ along the imaginary axis, which is the amplitude response. To obtain an appropriate scale, change the arguments in the **meshgrid** command to **-4:0.05:0, -2:0.05:2** and the input vector in the **axis** command to **[-2 2 -4 0 0 6]**. Also, to enhance the capability of viewing the frequency response and poles, change the surface view angle by using the statement **view(-5,60)**; rather than **view(-24,40)**.