EE 216 - Experiment 3
Superposition Integral Evaluation
(Convolution Evaluation)

Objectives:

To learn to use MATLAB to evaluate a convolution integral and apply it to system analysis.

Theory:

The convolution integral is defined to be (3.1)

$$f_c(t) = \int_{-\infty}^{\infty} f_1(\tau) f$$

We say that $f_c(t)$ is the continuous-time convolution (or continuous convolution) of the two *functions* $f_1(t)$ and $f_2(t)$ and often use the short notation,

$$f_c(t) = f_1(t) * f_2(t) \tag{3.2}$$

Convolution is commutative, associative and distributive.

If $f_1(t)$ equals a system input $x(t)$ and $f_2(t)$ equals the system impulse response *h*(*t*), then $f_c(t)$ equals the system output $y(t)$, if the system is linear, time-invariant, and has zero initial conditions. This integral is called the superposition integral.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \tag{3.3}$$

Since

$$f_c(t) = \lim_{T \to 0} \sum_{m=-\infty}^{\infty} f_1(mT) f_2(t - mT) T$$

(3.4)

then an approximation to the convolution value at $t = nT$ is

$$f_c(nT) \doteq T \sum_{m=-\infty}^{\infty} f_1(mT) f_2((n-m)T) \tag{3.5}$$

Note that the samples of $f_1(t)$, $f_2(t)$ and $f_c(t)$ all have the same spacing, *T*. The approximation becomes better as the function sample spacing *T* is decreased. We can evaluate eq. (3.5) with a computer for finite-length intervals of the indices *n* and *m*. That is, we can evaluate

$$f_c(nT) \doteq T \sum_{m=a_1}^{b_1} f_1(mT)\, f_2((n-m)T) \qquad a_c \le n \le b_c \qquad (3.6)$$

where the samples of $f_1(t)$ extend from $f_1(a_1T)$ to $f_1(b_1T)$. If the samples of $f_2(t)$ extend from $f_2(a_2T)$ to $f_2(b_2T)$, then the samples of $f_c(t)$ extend from $f_a(a_cT) = f_c((a_1 + a_2)T)$ to $f_c(b_cT) = f_c((b_1 + b_2)T)$. Note that there are $k_1 = b_1 - a_1 + 1$ samples of $f_1(t)$, $k_2 = b_2 - a_2 + 1$ samples of $f_2(t)$, and $k_c = b_c - a_c + 1 = (b_1 - a_1) + (b_2 - a_2) + 1 = k_1 + k_2 - 1$ samples of $f_c(t)$.

The MATLAB function **z=conv(x,y)** computes the summation specified in eq. (3.6). Thus, we can compute approximate values for the samples of $f_c(t)$ with the statement **fc=T*conv(f1,f2)**. (Note that * in MATLAB means multiplication, not convolution.) The first samples in the three row-arrays are **f1(1)** = $f_1(a_1T)$, **f2(1)=** $f_2(a_2T)$, and **fc(1)** = $f_c(a_cT) = f_c((a_1 + a_2)T)$. If the samples used for both $f_1(t)$ and $f_2(t)$ start at $t = 0$, then the first sample in all three row-arrays corresponds to the same time, which is, $t = 0$ This is not true unless. $a_1 = a_2 = 0$. If a_1 and a_2 are not both equal to zero, then the first sample in each row-array does not correspond to the same time. In this case, **f1(1)**, **f2(1)**, and **fc(1)** correspond to $t = a_1T$, $t = a_2T$, and $t = a_cT$, respectively. Thus, if the signals are to be plotted, then care must be taken to start them at the appropriate times. The easiest way to do this is to create for each function a separate time array that starts and ends at the appropriate time. These times are $t = a_1T$, and $t = b_1T$, for $f_1(t)$, $t = a_2T$ and $t = b_2T$ for $f_2(t)$, and $t = a_cT$ and $t = b_cT$ for $f_c(t)$. Note that the separation of time values for all three arrays must be the same.

If the signals $f_1(t)$ and $f_2(t)$ are zero outside of the time intervals $c_1T \le t \le d_1T$ and $c_2T \le t \le d_2T$, respectively, then $f_c(t)$ is zero outside of the time interval $(c_1 + c_2)T \le t \le (d_1 + d_2)T$. In this case it is preferable to make $a_1T = a_2T < \min\{c_1T,\ c_2T,\ (c_1 + c_2)T\}$ and $b_1T = b_2T > \max\{d_1T, d_2T, (d_1 + d_2)T\}$. In this way, we see the zero portions of all three signals if we plot them all from a_1T to b_1T. It is possible, though not common, that none of the function samples taken or computed overlap on the time axis. For example, if $a_1 = 100$, $b_1 = 150$, $a_2 = 200$, and $b_2 = 250$, then $a_c = 300$ and $b_c = 400$ and $f_1(nT)$ extends for $100T \le t \le 150T$, $f_2(nT)$ extends for $200T \le t \le 250T$, and $f_c(nt)$ extends for $300T \le t \le 400T$. The time scale on a single plot of all three functions must extend from 100T to 400T.

We have indicated that the accuracy of a continuous convolution computed with **fc=T*conv(f1,f2)** improves as T becomes smaller. If the original functions do not go to zero outside some time interval or are so long that only a portion of them can be used, then there is another accuracy problem. This problem is caused by the fact that the sum should extend from $-\infty$ to $+\infty$ but we can only use a finite number of terms. If the functions are very small outside some time interval, then the inaccuracy will be small if only the very small values are eliminated when choosing the signal portion used (see Laboratory Procedure Part 2). Assume that we use only a portion of a function that does not go to zero outside a time interval. We then convolve this function portion with a second function that is nonzero only on an interval that is much shorter than the portion of the first function used. In this case, the inaccuracy is limited to the ends of the convolution result. It is only at these ends that we do not have complete overlap of the functions being convolved (see Laboratory Procedure Part 3). What if the second function does not go to zero outside the short interval but becomes small? In this case, most of the inaccuracy still occurs at the ends of the convolution result but there is some inaccuracy throughout since the short signal is truncated.

Preliminary :

1. Find appropriate plotting scales for parts 1 and 3. (a plotting scale consists of the values of the x and y axes.) You can also specify the x-axis scale as your time variable, t. Be sure to show all significant features of the waveform.

Laboratory Procedure:

1. Use **conv** to compute samples of $z(t) = x(t) * y(t)$ when

   $$x(t) = -(t + 0.25[u(t+2) - u(t-1)]$$

   and

   $$y(t) = 2e^{-0.4(t+0.5)} \cos 0.5\pi t - \frac{\pi}{4}\left[u(t+0.5) - u(t-2.5)\right]$$

   Note that $x(t)$ can be nonzero from -2 to 1 in t and $y(t)$ extends from -0.5 to 2.5. Therefore, $z(t)$ extends from -2+(-0.5)=-2.5 to 1+2.5 =3.5 Choose samples of $x(t)$ and $y(t)$ over the interval $-4 \le t \le 4$ so that all three functions will show completely on plots over this interval $\left[-4, \min(-2,-0.5,-25) \text{ and } 4 > \max\ \left(1,2.5,3.5\right)\right]$. Use a sample spacing of T =0.05. What range of time do the samples computed for $z(t)$ encompass? What is the length of the non-zero portions of $x(t)$, $y(t)$, and $z(t)$? Plot $x(t)$, $y(t)$, and $z(t)$ on the same set of axes for $-4 \le t \le 4$. Try some additional values of T, both larger and smaller, and comment on the results.

   You will investigate the effect of signal truncation in Parts 2 and 3. Thus, the required MATLAB script is similar and you should construct an m-file containing the script for Part 2 so you can modify it for Part 3.

2. The convolution of $x(t) = e^{-0.5t}u(t)$ and $y(t) = 2e^{0.75t}u(-t)$ is

$$z(t) = \int_{-\infty}^{\infty} 2e^{-0.5w}u(w)e^{0.75(t-w)}u(w-t)dw$$

$$= 2e^{0.75t}\int_{a}^{\infty} e^{-1.25w}\, dw \begin{cases} a=0 & t<0 \\ a=t & t\geq 0 \end{cases}$$

$$= \begin{cases} 1.6e^{0.75t} & t<0 \\ 1.6e^{-0.5t} & t\geq 0 \end{cases}$$

Choose segments of $x(t)$ and $y(t)$ that extend over the interval, $-8 \leq t \leq 8$. Use **conv** with $T = 0.01$ to compute samples of $z(t)$. Plot the approximation defined by these samples of $z(t)$. Construct the plot for the full interval of time for which samples of $z(t)$ are computed. Also plot the analytically derived mathematical expression for $z(t)$ on the same set of axes. Plot $x(t)$ and $y(t)$ on a second set of axes having the same time scale. Repeat all of the above for signal segments extending over $-6 \leq t \leq 6$, $-4 \leq t \leq 4$, and $-2 \leq t \leq 2$ to observe and comment on the effect of truncating nonsmall signal values.

3. The convolution of $x(t) = 3\Pi\left(\left(t-0.875\right)/1.75\right)$ and $y(t) = 1.5\sin\left(4\pi t\right)$ is

$$z(t) = \int_{-\infty}^{\infty} 4.5\Pi\left(\left(w-0.875\right)/1.75\right)\sin\left(4\pi\left(t-w\right)\right)dw$$

$$= 4.5\int_{0}^{1.75}\sin\left(4\pi\left(t-w\right)\right)dw = -\frac{2.25}{\pi}\cos\left(4\pi t\right)$$

$\pi(t)$ is a rectangular pulse defined as $u\left(t+\frac{1}{2}\right)-u\left(t-\frac{1}{2}\right)$. Choose segments $x(t)$ and $y(t)$ that extend over the interval $-2 \leq t \leq 3$. Use **conv** with $T = 0.005$ to compute samples of $z(t)$. Plot this approximation to $z(t)$ and the analytically derived mathematical expression for $z(t)$ on the same set of axes. Use a time scale that covers the full interval of time for which samples of z(t) were computed. Plot $x(t)$ and $y(t)$ on a second set of axes having the same time scale. Comment on the accuracy of the approximation based on observations of the plotted functions.

4. Find samples of the impulse response of the system given in Experiment 2 Part 6. (Hint: Use the system equation you calculated in the Experiment 2 preliminary but with an impulse as the input and zero initial conditions. You can also reuse the script of Experiment 2, Part 6.) Use a sample spacing of 0.01 and $-1 \leq t \leq 12$. Then use **conv** to evaluate the superposition integral and find the filter output when the input signal $x(t) = 2-\cos\left(0.2\pi t\right)+0.25\cos\left(2\pi t\right)$ and the initial conditions are zero. Plot the impulse response on one set of axes and the input and output signal on a second set of axes. Use the time scale $-12 \leq t \leq 12$ for both sets of axes. Compute sufficient data for the output so you can eliminate end effect inaccuracies in your output signal plot. Discuss the effect of the system on the three components of the signal.